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A general method for constructing Timoshenko-type theories☆

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ABSTRACT

The equations of the plane theory of for the elasticity bending of a long strip are reduced by the method of simple iterations to the solution of a system of two equations for the displacement of the axis of the strip and the shear stress. If the transverse load varies slowly along the strip, the resolvent equations reduce to a single equation that is identical to the classical equation for the bend of a beam. When a local load is applied, the resolvent equation acquires an additional singular term that is the solution of the equation for the shear stresses under the assumption that the displacement (deflection) is a function of small variability. The convergence of the solution in an asymptotic sense is demonstrated. The application of the method of simple iterations to the dynamic equations for the bending of a strip also leads to a system of two resolvent equations in the displacement of the axis of the strip and the shear stress. These equations reduce to a single equation that is identical with the well-known Timoshenko equation. Hence, the procedure for using the method of simple iterations that has been developed can be classified as a general method for obtaining Timoshenko-type theories. An equation is derived for the bending of a strip on an elastic base with an isolated functional singular part with two bed coefficients, corresponding to the transverse and longitudinal springiness of the base.

1. Introduction

Simplified models can be constructed by discarding secondary effects in the more "exact" models such as when changing from spatial theories to two- or one-dimensional theories. However, in many cases, such as in the theory of shells, for example, the problem of neglecting secondary in the three-dimensional equations is not simple and the essence of the investigation lies in evaluating the accuracy of the different two-dimensional models, the number of which, generally speaking, is large compared with the initial three-dimensional model. It has been found[1] that exciting theories can be constructed using a single approach, the method of simple iterations. In this case, the asymptotic estimates (and asymptotic integration) accompanying the simple iterations enable-one to discard the minor terms with respect to the principal terms, to estimate the rate of convergence of the iterative process, to prove asymptotic convergence and to obtain simple models of phenomena. All the simplified models (submodels of the theory of elasticity) can deviously be constructed by the method of simple iterations (the SI method) and an asymptotic estimate of their accuracy can be obtained by the method of asymptotic integration (the AI method) (we will henceforth refer to the combination of the method of simple iteration and the method of asymptotic integration as the SIAI method). In other words, if this position is correct, contraction the mapping principle provides the foundation of each of the above-mentioned methods. Naturally, this method is explained by considering simpler problems.

In courses on the theory of elasticity, the solution of a plane problem for a long strip using a polynomial representation of the stress function under the implicit assumption that only a slowly varying stress-strain state exists is given instead of the theory of a beam (see Ref. 2, for example). In this paper, the application of the procedure, developed earlier for equations in the shell theory,[1] to the equations of the plane problem of the theory of elasticity for a strip led to the construction of a new theory of a beam, which has already been previously discussed.[3] Application of the same procedure to the dynamical equations of the plane problem for a strip leads to the construction of an asymptotic theory of the oscillations of the strip, the resolving equation of which is identical with the well-known Timoshenko equation.[4] This enables one to assume that the proposed procedure can be used to construct models of the Timoshenko-type in the case of static and dynamic problems for beams, plates and shells.

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In handbooks on the strength of materials, the action of local loads on a beam is taken into account in the following way. A load, distributed over a small interval of length, is replaced by the statically equivalent concentrated force, considered as a discontinuity in the transverse force. In the case of dynamic action, the load is expanded in series in the natural modes of oscillation of the beam. On the basis of calculated and experimental data, it is considered that, in the case of short beams and local loads, better results are obtained using the Timoshenko equation, in which the corrections associated with the rotational inertia and the transverse shear of an element of the beam length are taken into account[4]:

$$EI\frac{\partial^4 y^*}{\partial x^{*4}} - \rho I\left(1+\frac{E}{k'G}\right)\frac{\partial^4 y^*}{\partial x^{*2}\partial t^{*2}} + \frac{\rho^2 I}{k'G}\frac{\partial^4 y^*}{\partial t^{*4}} + \rho F\frac{\partial^2 y^*}{\partial t^{*2}} = 0 \quad (1.1)$$

Here, E is Young's modulus, G is the shear modulus, k' is a coefficient which depends on the shape of the cross section, F is the cross-section area, ρ is the mass per unit length of the beam, y^* is the deflection, x^* is the coordinate of the beam axis, t^* is the time, and I is the moment of inertia of the cross-section. There are different recommendations regarding the choice of he magnitude of the coefficient k': a value of 0.833 (Ref. 4) and, also, a value of 2/3 (Ref. 5) is used for a rectangular cross-section and a value of 3/4 for a circular cross-section; a value of $\pi^2/12$ has been suggested but values of 5/6 and $5/(6-\nu)$ (Ref. 6) have also been assumed.

If the differential operator with respect to time is equal to zero in Eq. (1.1), the Timoshenko equation is transformed into the classical equation for the bending of a beam. This implies that the intuitive Timoshenko theory only has a meaning when applied to dynamic problems.

From the position of the SIAI method, models of static and dynamic problems are obtained by a single route. According to the method of simple iterations, the values of an initial approximation are specified. These basic values are introduced into the equations as known values and the remaining required unknowns of the problem are calculated in terms of them. It can be assumed that setting the values of the initial approximation corresponds to the introduction of a hypothesis of the type of a straight and undeformable normal in the theory of thin-wall systems, usually associated with the semi-inverse Saint-Venant method.[7] In the SIAI method, a sequence of calculations of the unknowns is constructed such that the unknowns specified in the zeroth approximation, which are calculated in the first approximation, have a small correction $O(\varepsilon^k)$, where $\varepsilon = h/l$ is a small parameter, $2h$ is the height of the beam (strip), l is its length and k is a positive number determining the asymptotic of the solution. This enables us to speak of the convergence of the method in an asymptotic sense (limiting when $\varepsilon \to 0$). If there is a sequence of applied contraction operators, the required unknown calculated in the first approximation and the specified unknown in the first approximation when the boundary conditions are satisfied give a model with a dimensional that one less than that of the initial model. The contraction property is proced using the AI method.

2. Static bending of a strip

We refer a long rectangular strip to the rectangular system of coordinates x^*, z^* such that $0 \le x^* \le l$, $-h \le z^* \le h$. The strip is bent under the load $q^* = q^*(x)$ distributed over the upper edge and directed downwards.

We introduce the dimensionless coordinates $x = x^*/l$ and $z = z^*/h$, the dimensionless displacements $u = u^*/h$ and $w = w*$ along the x^* and z^* axes respectively and the dimensionless stresses $\sigma_x = \sigma_x^*/E$, $\sigma_z = \sigma_z^*/E$ and $\tau = \tau^*/E$ (dimensioned displacements, stresses and loads are denoted by an asterisk).

We will take the equations of the phase problem in the theory of elasticity, which describe the stress-strain state of a strip of unit width in the form

$$\frac{\partial u}{\partial z} = -\varepsilon\frac{\partial w}{\partial x} + 2(1+\nu)\tau, \quad \frac{\partial \sigma_z}{\partial z} = -\varepsilon\frac{\partial \tau}{\partial x}, \quad \varepsilon_x = \varepsilon\frac{\partial u}{\partial x}$$

$$\sigma_x = \varepsilon_x + \nu\sigma_z, \quad \varepsilon_z = (1-\nu^2)\sigma_z - \nu\varepsilon_x, \quad \frac{\partial w}{\partial z} = -\varepsilon\frac{\partial \sigma_x}{\partial x} \quad (2.1)$$

Here, $\varepsilon = h/l$ is a small parameter.

We shall seek the solution of the system as follows. Putting

$$w = w_0(x), \quad \tau = \tau_0(x) \quad (2.2)$$

in the first and second equations as the known values of the zeroth approximation, we calculate u_0 and σ_{z0}. We then calculate ε_{x0} and find σ_{x0} and ε_{x0} using the elasticity relations. On substituting them into the last two equations of the system, we obtain w_1 and τ_1 in the first approximation. The process of calculating the following approximations can be continued.

For convenience in the analysis, we will expand the initial approximation (2.2) into three parts

$$w = w_0(x), \quad \tau = \tau_0 = 0 \quad (2.3)$$

$$w = w_0 = 0, \quad \tau = \tau_0(x) \quad (2.4)$$

$$w = w_0 = 0, \quad \tau = \tau_0 = 0 \quad (2.5)$$

We will refer to the process of calculating the required unknowns, starting out from relations (2.3), (2.4) and (2.5), as the w-process, the τ-process and the 0-process respectively. In the first process, all the required unknowns will be expressed in terms of the value of the initial approximation w_0 as particular solutions of problem (2.1) for the set values (2.3). In the second process, they will similarly be expressed in terms of τ_0 and, in the third process, in terms of the arbitrary functions of integration of the problem that is homogeneous in the values of the initial approximation.

Calculation of the components of the stress-strain state gives:in the w-process

$$w = w_0(x), \quad \tau_0 = 0, \quad u_0 = -\varepsilon w_0' z$$
$$\sigma_{z0} = 0, \quad \varepsilon_{x0} = -\varepsilon^2 w_0'' z, \quad \sigma_{x0} = -\varepsilon^2 w_0'' z$$
$$\varepsilon_{z0} = \nu\varepsilon^2 w_0'' z, \quad w_1 = \nu\varepsilon^2 w_0'' z^2/2, \quad \tau_1 = \varepsilon^2 w_0''' z^2/2 \tag{2.6}$$

in the τ-process

$$w_0 = 0, \quad \tau = \tau_0(x), \quad u_0 = 2(1+\nu)\tau_0 z, \quad \sigma_{z0} = -\varepsilon\tau_0' z$$
$$\varepsilon_{x0} = 2(1+\nu)\varepsilon\tau_0' z, \quad \sigma_{x0} = (2+\nu)\varepsilon\tau_0' z, \quad \varepsilon_{z0} = -(1+\nu)^2\varepsilon\tau_0' z$$
$$w_1 = -(1+\nu)^2\varepsilon\tau_0' z^2/2, \quad \tau_1 = -(2+\nu)\varepsilon^2\tau_0'' z^2/2 \tag{2.7}$$

in the 0-process

$$w_0 = 0, \ \tau_0 = 0, \ u_0 = u_0(x), \ \sigma_{z0} = \sigma_{z0}(x), \ \varepsilon_{x0} = \varepsilon u_0'$$
$$\sigma_{x0} = \varepsilon u_0' + \nu\sigma_{z0}, \quad \varepsilon_{z0} = (1-\nu^2)\sigma_{z0} - \nu\varepsilon u_0'$$
$$w_1 = [(1-\nu^2)\sigma_{z0} - \nu\varepsilon u_0']z, \quad \tau_1 = (-\varepsilon^2 u_0'' - \nu\varepsilon\sigma_{z0}')z \tag{2.8}$$

Differentiation with respect to x is denoted by a prime. The arbitrary functions of integration u_0 and z_0 depend solely on x.

The conditions $\tau* = 0$, $\sigma_z^* = -q*$ must be satisfied on the side of the strip $z^* = -h$. There are no stresses: $\tau* = \sigma_z^* = 0$ on the side $z^* = h$. In dimensionless form, these conditions are written as

$$\tau = \sigma_z = 0 \text{ when } z = -1, \quad \tau = 0, \ \sigma_z = -q \text{ when } z = 1; \quad q = q^*/E \tag{2.9}$$

The boundary conditions will be satisfied by the values of the first approximation (2.6)–(2.8). The stress τ has the form

$$\tau = [\varepsilon^3 w_0''' - (2+\nu)\varepsilon^2\tau_0'']z^2/2 + \tau_0 - (\varepsilon^2 u_0'' + \nu\varepsilon\sigma_{z0}')z \tag{2.10}$$

The stress σ_z is calculated using the second equation of system (2.1) and expressions (2.6)–(2.8)

$$\sigma_z = -[\varepsilon^4 w_0^{1V} - (2+\nu)\varepsilon^3\tau_0''']z^3/6 - \varepsilon\tau_0' z + (\varepsilon^3 u_0''' + \nu\varepsilon^2\sigma_{z0}'')z^2/2 + \sigma_{z0} \tag{2.11}$$

After substituting these stresses into the boundary conditions (2.9), we obtain four equations for determining the functions w_0, τ_0, u_0, σ_{z0}. It can be shown that the equations for determining u_0, σ_{z0} separate:

$$\varepsilon^2 u_0'' + \nu\varepsilon\sigma_{z0}' = 0, \quad \varepsilon^3 u_0''' + \nu\varepsilon^2\sigma_{z0}'' + 2\sigma_{z0} = -q$$

and their solutions can be written as

$$\sigma_{z0} = -\frac{q}{2}, \quad u_0 = u_0^p + u_0^g; \quad u_0^p = \frac{\nu}{2\varepsilon}\int q\,dx, \quad u_0^g = C_1 x + C_2, \quad C_i = \text{const} \tag{2.12}$$

To the equations for determining of τ_0 and w_0, we give the form

$$\varepsilon^3 w_0''' - (2+\nu)\varepsilon^2\tau_0'' + 2\tau_0 = 0, \quad \varepsilon^4 w_0^{1V} - (2+\nu)\varepsilon^3\tau_0''' + 6\varepsilon\tau_0' = 3q \tag{2.13}$$

By calculating from them that

$$\varepsilon\tau_0' = \frac{3}{4}q \tag{2.14}$$

and substituting this expression into the second equation, we obtain

$$\frac{2}{3}\varepsilon^4 w_0^{1V} = -q + \frac{2+\nu}{2}\varepsilon^2 q'' \tag{2.15}$$

If the treatment is confined to slowly varying loads, assuming that $\varepsilon^2 q'' \sim \varepsilon^2 q$, then the second term on the right-hand side of this equality can be neglected and the resolvent equation for the strip obtained in this manner is identical with the classical equation for the bending of a beam.

Eq. (2.13) can be looked at in a different way. We assume that the function w_0 and its derivatives are functions of zero variability, that is, application of the operator $\partial/\partial x$ in an asymptotic sense is equivalent to the preservation of the differentiable function in its previous form, that is, to multiplying it by ε^0. According to this, we represent the final solution of the first equation of (2.13), written in the unknown τ_0

$$\varepsilon^2\tau_0'' - k^2\tau_0 = \varepsilon^3 k^2 w_0'''/2, \quad k^2 = 2/(2+\nu) \tag{2.16}$$

and having a small parameter accompanying the derivative in the form of the sum of the particular solution $\tau_0^p = -\varepsilon^3 w_0'''/2$ and the general solution

$$\tau_0^g = \begin{cases} C_1 e^{k(x-c)/\varepsilon} & \text{when } x \le c \\ C_2 e^{-k(x-c)/\varepsilon} & \text{when } x \ge c \end{cases}$$

referred to a certain perturbation point $x = c$, in the neighbourhood of which a stress state of the boundary-layer type arises.

Substituting the sum $\tau_0^p = -\varepsilon^3 w_0'''/2 + \tau_0^g$ into Eq. (2.14), we obtain the equation

$$\frac{2}{3}\varepsilon^4 w_0^{IV} - \frac{4}{3}\varepsilon \tau_0^{g\prime} = -q \tag{2.17}$$

The dynamic analogues of Eqs (2.15) and (2.17) will be obtained below. At the same time, if

$$q = \begin{cases} B_1 e^{k(x-b)/\varepsilon} & \text{when } x \le b \\ B_2 e^{-k(x-b)/\varepsilon} & \text{when } x \ge b \end{cases} \tag{2.18}$$

where the point b, which determines the locus of the application of the load, generally speaking, coincides with the point c, the right-hand side of Eq. (2.15) vanishes and the solution of the homogeneous equation has the form

$$w_0 = a_0 + a_1 x + a_2 x^2 + a_3 x^3; \quad a_i = \text{const} \tag{2.19}$$

confirming the small variability of the function w_0. The constants B_1 and B_2 characterize the intensity of the local load applied, and it is necessary to consider them as being given. Putting $B_1 = B_2 = Pk\varepsilon^{-1}/2$ (P = const) in relations (2.18), we see that

$$\lim_{\varepsilon \to 0} q = P\delta(x - b)$$

The magnitude of P determines the intensity of the applied force.

3. Partitioning of the boundary conditions on the short sides and the continuity conditions

We will now consider a procedure for applying the relations obtained to the problem of the bending of a strip by a force P, applied at a point with coordinates $x = c$, $z = 1$. We shall use the asymptotic representation of a concentrated force by expression (2.18) in which $b = c$ and $B_1 = B_2 = Pk\varepsilon^{-1}/2$. To be specific, we will assume that the ends of the strip at $x = 0$ and $x = 1$ are rigidly clamped, that is, at the ends $u = w = 0$. In order to satisfy the boundary conditions and the continuity conditions on the line $x = c$, which the displacements and stresses must satisfy, we set up, using the elementary solutions (2.6)–(2.8), the necessary expressions for u, w, σ_x, τ in the first approximation

$$\begin{aligned} u &= -\varepsilon w_0' z + \varepsilon^\alpha 2(1+\nu)\tau_0 z + u_0 \\ w &= w_0 + \nu\varepsilon^2 w_0'' z^2/2 - \varepsilon^{\alpha+1}(1+\nu)^2 \tau_0' z^2/2 + [(1-\nu^2)\sigma_{z0} - \nu\varepsilon u_0']z \\ \sigma_x &= [-\varepsilon^2 w_0'' + \varepsilon^{\alpha+1}(2+\nu)\tau_0']z + \varepsilon u_0' + \nu\sigma_{z0} \\ \tau &= \varepsilon^3 w_0''' z^2/2 + \varepsilon^\alpha[\tau_0 - (2+\nu)\varepsilon^2 \tau_0'' z^2/2] - (\varepsilon^2 u_0'' + \nu\varepsilon\sigma_{z0}')z \end{aligned} \tag{3.1}$$

Here, the basic quantity τ_0 is multiplied by ε^α with an undetermined exponent α that will be determined from the condition for the equations for the arbitrarinesses of integration corresponding to the boundary conditions and matching conditions to be solvable. Multiplication is possible since τ_0 is a solution of a homogeneous equation. In the second expression, the second term can be neglected compared with the first as a quantity of the second order of smallness. It is seen that the equations for determining of w_0, τ_0 and u_0, σ_{z0} decompose. The matching equations for determining the last two basic quantities do not contain discontinuous functions and are satisfied identically. The boundary conditions when $x = 0$ and $x = 1$, and the continuity condition on the line $z = c$ are now written in the form:for the functions w_0 and τ_0

$$-\varepsilon w_0' + \varepsilon^\alpha 2(1+\nu)\tau_0 = 0, \quad w_0 - \varepsilon^{\alpha+1}(1+\nu)^2 \tau_0' z^2/2 = 0 \tag{3.2}$$

$$\begin{aligned} &\{-\varepsilon w_0' + \varepsilon^\alpha 2(1+\nu)\tau_0\} = 0, \quad \{w_0 - \varepsilon^{\alpha+1}(1+\nu)^2 \tau_0' z^2/2\} = 0 \\ &\{-\varepsilon^2 w_0'' z + \varepsilon^{\alpha+1}(2+\nu)\tau_0' z\} = 0, \quad \{\varepsilon^3 w_0''' z^2/2 + \varepsilon^\alpha[\tau_0 - (2+\nu)\varepsilon^2 \tau_0'' z^2/2]\} = 0 \end{aligned} \tag{3.3}$$

for the functions u_0 and σ_{z0}

$$u_0 = 0, \quad (1-\nu^2)\sigma_{z0} - \nu\varepsilon u_0' = 0 \tag{3.4}$$

$$\{u_0\} = 0, \quad \{(1-\nu^2)\sigma_{z0} - \nu\varepsilon u_0'\} = 0, \quad \{\varepsilon u_0' + \nu\sigma_{z0}\} = 0, \quad \{\varepsilon^2 u_0'' + \nu\sigma_{z0}'\} = 0 \tag{3.5}$$

The braces denote that it is necessary to write the expression in them for $x \le c$ and $x \ge c$ and to subtract the first value from the second on the line $x = c$.

In relations (3.2) and (3.3), we choose

$$\alpha = 3 \tag{3.6}$$

and only leave the leading terms in the matching conditions. Taking account of Eq. (2.16) in the last of them, the matching conditions (3.3) can then be written in the form

$$\{w_0'\} = 0, \{w_0\} = 0, \; \{w_0''\} = 0, \; \{\varepsilon^3 w_0'''/2 + \varepsilon^3 \tau_0^g\} = 0 \tag{3.7}$$

Hence, for the solution w_0 of the form of (2.19), where the representations of the solution for the left-hand and right-hand sides of the strip with respect to the point $x = c$ each contain four constants, there are four conditions (3.2) at the ends and four matching conditions (3.3).

By virtue of relation (2.12), the stress σ_{z0} is negligibly small in the boundary conditions (3.4) at the ends of the strip, and the condition reduces to the requirement that there is no transverse deformation on account of the Poisson effect $\nu\varepsilon_x = \nu\varepsilon u'_0$ at the clamping position. It is impossible to satisfy this condition within the framework of the approximation (2.2) being considered. For this, it is necessary to abandon the constraint that τ depends solely on x and does not depend on z (Ref. 3).

In the continuity conditions (3.5), the last condition is a consequence of the penultimate condition and can be discarded. We now note that the stress σ_{z0} is continuous at the point $x = c$ and all the conditions reduce to the two conditions:

$$\{u_0\} = 0, \quad \{u_0'\} = 0 \tag{3.8}$$

The functions w_0 and τ_0, which satisfy conditions (3.7), have the form

$$\frac{8\varepsilon^4}{3P} w_0 = \begin{cases} -(c - 2c^2 + c^3)x^2 + (1/3 - c^2 + 2/3c^3)x^3 & \text{when } x \le c \\ c^3/3 - c^2 x + (2c^2 - c^3)x^2 - (c^2 - 2/3c^3)x^3 & \text{when } x \ge c \end{cases}$$

$$\frac{8\varepsilon}{3P}\tau_0^g = \begin{cases} -e^{k(x-c)/\varepsilon} & \text{when } x \le c \\ e^{-k(x-c)/\varepsilon} & \text{when } x \ge c \end{cases}$$

Substituting the now known quantities w_0 and τ_0 into the expression, for example, for σ_x from relations (3.1), we obtain

$$\frac{4\varepsilon^2}{3Pz}\sigma_x = \begin{cases} c(1-c)^2 - (1 - 3c^2 + 2c^3)x \\ -c^2(2-c) + c^2(3-2c)x \end{cases} + \varepsilon\frac{(2+\nu)k}{4} \times \begin{cases} e^{k(x-b)/\varepsilon} \\ e^{-k(x-b)/\varepsilon} \end{cases}$$

The upper value is taken when $x \le c$ and the lower value when $x \ge c$.

The function u_0 satisfying conditions (3.8) has the form

$$\frac{4\varepsilon}{\nu P}u_0 = \begin{cases} e^{k(x-c)/\varepsilon} - 2x & \text{when } x \le c \\ -e^{-k(x-c)/\varepsilon} - 2x + 2 & \text{when } x \ge c \end{cases}$$

4. The dynamic bend of a strip

We will now use the procedure described above to construct a dynamic submodel of the oscillations of a long strip. Adding the dimensionless inertial terms to the static bending equations (2.1), we obtain the following equations of motion

$$\frac{\partial u}{\partial z} = -\varepsilon\frac{\partial w}{\partial x} + 2(1+\nu)\tau, \quad \frac{\partial \sigma_z}{\partial z} = -\varepsilon\frac{\partial \tau}{\partial x} + p^2 w$$

$$\varepsilon_x = \varepsilon\frac{\partial u}{\partial x}, \quad \sigma_x = \varepsilon_x + \nu\sigma_z, \quad \varepsilon_z = (1-\nu^2)\sigma_z - \nu\varepsilon_x$$

$$\frac{\partial w}{\partial z} = \varepsilon_z, \quad \frac{\partial \tau}{\partial z} = -\varepsilon\frac{\partial \sigma_x}{\partial x} + p^2 u \tag{4.1}$$

The dimensionless operator

$$p^2 = \frac{\rho h^2}{ET^2}\frac{\partial^2}{\partial t^2}$$

has been introduced here, in which ρ is the specific density of the strip material, t^* is the dimensional time, $t = t^*/T$ is the dimensionless time and T is a certain characteristic period of the oscillations of a strip of unit width, length l and height h.

As in the static problem, setting the initial approximations by means of (2.3)–(2.5), we calculate the remaining unknowns in the zeroth approximation, and w and τ, which had earlier been assigned in the zeroth approximation, in the first approximation. This gives;

in the w-process

$$w = w_0(x), \quad \tau_0 = 0, \quad u_0 = -\varepsilon w_0' z, \quad \sigma_{z0} = p^2 w_0 z, \quad \varepsilon_{x0} = -\varepsilon^2 w_0'' z$$

$$\sigma_{x0} = (-\varepsilon^2 w_0'' + \nu p^2 w_0)z, \quad \varepsilon_{z0} = \lfloor (1-\nu^2)p^2 w_0 + \nu\varepsilon^2 w_0'' \rfloor z$$

$$w_1 = \lfloor (1-\nu^2)p^2 w_0 + \nu\varepsilon^2 w_0'' \rfloor z^2/2, \quad \tau_1 = \lfloor \varepsilon^3 w_0''' - (1+\nu)\varepsilon p^2 w_0' \rfloor z^2/2$$

in the τ-process

$$w_0 = 0, \quad \tau = \tau_0(x), \quad u_0 = 2(1+\nu)\tau_0 z, \quad \sigma_{z0} = -\varepsilon\tau_0' z$$

$$\varepsilon_{x0} = 2(1+\nu)\varepsilon\tau_0' z, \quad \sigma_{x0} = (2+\nu)\varepsilon\tau_0' z, \quad \varepsilon_{z0} = -(1+\nu)^2\varepsilon\tau_0' z$$

$$w_1 = -(1+\nu)^2\varepsilon\tau_0' z^2/2, \quad \tau_1 = [-(2+\nu)\varepsilon^2\tau_0'' + 2(1+\nu)p^2\tau_0]z^2/2$$

in the 0-process

$$w_0 = 0, \quad \tau_0 = 0, \quad u_0 = u_0(x), \quad \sigma_{z0} = \sigma_{z0}(x), \quad \varepsilon_{x0} = \varepsilon u_0'$$

$$\sigma_{x0} = \varepsilon u_0' + \nu\sigma_{z0}, \quad \varepsilon_{z0} = (1-\nu^2)\sigma_{z0} - \nu\varepsilon u_0'$$

$$w_1 = [(1-\nu^2)\sigma_{z0} - \nu\varepsilon u_0'], \quad \tau_1 = (-\varepsilon^2 u_0'' - \nu\varepsilon\sigma_{z0}' + p^2 u_0)z$$

Expressions for the stresses τ_1 and σ_{z1}, calculated in the first approximation, can now be written as the sum of the elementary solutions obtained in the w-, τ- and 0-processes.

$$\tau = \lfloor \varepsilon^3 w_0''' - (1+\nu)\varepsilon p^2 w_0' - (2+\nu)\varepsilon^2\tau_0'' + 2(1+\nu)p^2\tau_0 \rfloor z^2/2 +$$

$$+ (-\varepsilon^2 u_0'' - \nu\varepsilon\sigma_{z0}' + p^2 u_0)z + \tau_0$$

$$\sigma_z = -\lfloor \varepsilon^4 w_0^{IV} - (1+\nu)\varepsilon^2 p^2 w_0'' - (2+\nu)\varepsilon^2\tau_0''' + 2(1+\nu)\varepsilon p^2\tau_0' \rfloor z^3/6 -$$

$$- (\varepsilon^3 u_0''' - \nu\varepsilon^2\sigma_{z0}'' + \varepsilon p^2 u_0')z^2/2 - \varepsilon\tau_0' z + p^2 w_0 z + \sigma_{z0}$$

Suppose a load $q^* = q^*(x^*, t)$, which can either be a distributed load or a local load, acts on the strip boundary $z^* = h$. In the first approximation, it is necessary to satisfy the conditions (in dimensionless form)

$$\tau = 0 \text{ when } z = \pm 1, \quad \sigma_z = -q(x,t) \text{ when } z = 1, \quad \sigma_z = 0 \text{ when } z = -1$$

After substituting the expressions written above for the stresses into these conditions, we obtain the equations for determining the required unknowns u_0, σ_{z0}, w_0, τ_0

$$\sigma_{z0} = -q/2, \quad -\varepsilon^2 u_0'' + p^2 u_0 = -\nu\varepsilon q'/2$$

$$\varepsilon^3 w_0''' - (1+\nu)\varepsilon p^2 w_0' - (2+\nu)\varepsilon^2\tau_0'' + 2(1+\nu)p^2\tau_0 + 2\tau_0 = 0$$

$$\varepsilon^4 w_0^{IV} - (1+\nu)\varepsilon^2 p^2 w_0'' - (2+\nu)\varepsilon^3\tau_0''' + 2(1+\nu)\varepsilon p^2\tau_0' - 6\varepsilon\tau_0' + 6p^2 w_0 = -3q \tag{4.2}$$

From the last two equations of (4.2), we find

$$\varepsilon\tau_0' = \frac{3}{4}(2p^2 w_0 + q) \tag{4.3}$$

and we substitute this expression into the last equation of (4.2). We obtain

$$\frac{2}{3}\left[\varepsilon^4 w_0^{1V} - \left(4 + \frac{5}{2}\nu\right)\varepsilon^2 p^2 w_0'' + 3(1+\nu)p^4 w_0\right] + 2p^2 w_0 = -q + \frac{2+\nu}{2}\varepsilon^2 q'' - (1+\nu)p^2 q \tag{4.4}$$

Eq. (1.1), reduced to dimensionless form,

$$\frac{2}{3}[\varepsilon^3 y^{1V} - (4+3\nu)\varepsilon^2 p^2 y'' + 3(1+\nu)p^4 y] + 2p^2 y = 0 \quad y = y^*/h$$

in the case of a beam of rectangular section ($I = 2bh^3/3$, $F = 2bh$) when $k' = 2/3$, as was recommended in Ref. 7, is essentially identical to Eq. (4.4) when $q = 0$. The difference is greater for the largest of the recommended values $k' = 5/6$.

Hence, the proposed procedure for constructing a submodel, as applied to the general dynamical equations of the theory of elasticity for long strip, give the Timoshenko model. This enables us to presume that the method of simple iterations generalizes the intuitive constructions of Timoshenko and makes the constructive building of models of the Timoshenko-type possible for other systems.

The last two equations of (4.2) are equivalent to Eq. (4.4) but are considerably more informative than it. The penultimate equation of (4.2), rewritten in the form

$$(2+\nu)\varepsilon^2\tau_0'' - 2(1+\nu)p^2\tau_0 - 2\tau_0 = \varepsilon^3 w_0''' - (1+\nu)\varepsilon p^2 w_0'$$

under certain conditions contains wave and singular solutions in τ_0. The time singularity is to be understood as a shock.

Assuming that the function w_0 is a function of zero variability in the coordinates and time, we can represent the solution of Eq. (4.5) in the form of a sum of the particular solution

$$\tau_0^p = [-\varepsilon^3 w_0''' + (1+\nu)\varepsilon p^2 w_0']/2$$

and the general solution τ_0^g. After substituting this sum into Eq. (4.3), we arrive at an equation of motion with an isolated singular part

$$\frac{2}{3}[\varepsilon^4 w_0^{1V} - (1+\nu)\varepsilon^4 p^2 w_0''] + 2\varepsilon^2 p^2 w_0 - \frac{4}{3}\varepsilon\tau_0^{g\prime} = -q$$

5. The static flexure of a strip on an elastic base

The conditions $\tau* = 0$, $\sigma_z^* = -q$ must be satisfied on the upper side of the strip $z^* = h$. The lower side of the strip is in contact with an elastic base in such a way that $\sigma_z^* = -q*$, $\tau* = k_2^* u*$. The coefficients of the bed k_1^* and k_2^* are assumed to be specified. In dimensionless form, these conditions are written as

$$\sigma_z = -q, \quad \tau = 0 \text{ when } z = 1 \text{ and } \sigma_z = k_1 w, \quad \tau = k_2 u \text{ when } z = -1 \tag{5.1}$$

The notation $k_1 = k_1^*/E$, $k_2 = k_2^*/E$ has been introduced here.

We shall satisfy the boundary conditions using the values of the first approximation (2.11) for σ_z, (2.10) for τ and (3.1) for w and u.

Substituting the expressions for the stresses and strains into boundary conditions (5.1), we obtain four equations for determining the four required quantities w_0, τ_0, u_0, σ_{z0}

$$-\lfloor \varepsilon^4 w_0^{1V} - (2+\nu)\varepsilon^3\tau_0''' \rfloor/6 - \varepsilon\tau_0' + (\varepsilon^3 u_0''' + \nu\varepsilon^2\sigma_{z0}'')/2 + \sigma_{z0} = -q$$

$$\lfloor \varepsilon^3 w_0''' - (2+\nu)\varepsilon^2\tau_0'' \rfloor/2 + \tau_0 - \varepsilon^2 u_0'' - \nu\varepsilon\sigma_{z0}' = 0$$

$$\lfloor \varepsilon^4 w_0^{1V} - (2+\nu)\varepsilon^3\tau_0''' \rfloor/6 + \varepsilon\tau_0' + (\varepsilon^3 u_0''' + \nu\varepsilon^2\sigma_{z0}'')/2 + \sigma_{z0} =$$
$$= k_1[w_0 + \nu\varepsilon^2 w_0''/2 - (1+\nu^2)\varepsilon\tau_0'/2 - (1-\nu^2)\sigma_{z0} + \nu\varepsilon u_0']$$

$$[\varepsilon^3 w_0''' - (2+\nu)\varepsilon^2\tau_0'']/2 + \tau_0 + \varepsilon^2 u_0'' + \nu\varepsilon\sigma_{z0}' = k_2[\varepsilon w_0' - 2(1+\nu)\tau_0 + u_0]$$

The quantity $\nu\varepsilon^2 w_0''/2$ on the right hand side of the third equation can be discarded as a quantity of the second order of smallness in ε compared with w_0. This quantity corresponds to the change in the height of the strip due to the fact that the longitudinal fibres of the strip are stretched (compressed) under flexure and thin down (thicken) due to the Poisson effect. By virtue of the fact that $k_1 \ll 1$,[8] the quantities $k_1(1-\varepsilon^2)\sigma_{z0}$ and $k_1(1+\nu)^2\varepsilon\tau'_0/2$ in the same place can be neglected compared the quantities σ_{z0} and $\varepsilon\tau'_0$ on the left-hand side respectively. By making use of the fact that $k_2 \ll 1$ in the last equation, the quantity $k_2 2(1+\nu)\tau_0$ on the right-hand side can be neglected compared with τ_0 on the left-hand side. We now turn our attention to the fact that the quantity u_0, introduced in the 0-process (2.8), is the elongation (shortening) of the middle line of the bar $z = 0$. This quantity is obviously far smaller than the transverse displacement of the middle line w_0 and the quantity $\nu\varepsilon u'_0$ on the right-hand side of the third equation can therefore be discarded compared with w_0. Finally, after the unimportant quantities have been discarded, the system takes the form

$$-\lfloor \varepsilon^4 w_0^{1V} - (2+\nu)\varepsilon^3\tau_0''' \rfloor/6 - \varepsilon\tau_0' + (\varepsilon^3 u_0''' + \nu\varepsilon^2\sigma_{z0}'')/2 + \sigma_{z0} = -q$$

$$\lfloor \varepsilon^4 w_0^{1V} - (2+\nu)\varepsilon^3\tau_0''' \rfloor/6 + \varepsilon\tau_0' + (\varepsilon^3 u_0''' + \nu\varepsilon^2\sigma_{z0}'')/2 + \sigma_{z0} = k_1 w_0$$

$$[\varepsilon^3 w_0''' - (2+\nu)\varepsilon^2\tau_0'']/2 + \tau_0 - \varepsilon^2 u_0'' - \nu\varepsilon\sigma_{z0}' = 0$$

$$[\varepsilon^3 w_0''' - (2+\nu)\varepsilon^2\tau_0'']/2 + \tau_0 + \varepsilon^2 u_0'' + \nu\varepsilon\sigma_{z0}' = k_2(\varepsilon w_0' + u_0)$$

Adding and subtracting the first two and the last two equations of the system, we reduce it to the form

$$\begin{aligned}
&\varepsilon^3 u_0''' + \nu\varepsilon^2\sigma_{z0}'' + 2\sigma_{z0} = -q + k_1 w_0\\
&-[\varepsilon^4 w_0^{IV} - (2+\nu)\varepsilon^3\tau_0''']/3 - 2\varepsilon\tau_0' = -q - k_1 w_0\\
&[\varepsilon^3 w_0''' - (2+\nu)\varepsilon^2\tau_0''] + 2\tau_0 = k_2(\varepsilon w_0' + u_0)\\
&2(\varepsilon^2 u_0'' + \nu\varepsilon\sigma_{z0}') = k_2(\varepsilon w_0' + u_0)
\end{aligned} \tag{5.2}$$

We multiply the third equation of the latter system by 1/3 and add it to the second. We obtain the relation

$$\varepsilon\tau_0' = \frac{3}{4}\left(q + k_1 w_0 - \frac{1}{3}k_2\varepsilon^2 w_0'' - \frac{1}{3}k_2\varepsilon u_0'\right)$$

It is seen from relations (2.8) that the quantity u_0 is the elongation of the middle line of the strip $z = 0$ and, correspondingly, the magnitude of $k_2\varepsilon u'_0/3$ is small compared with the value of $k_1 w_0$. We therefore finally have the relation

$$\varepsilon\tau_0' = \frac{3}{4}\left(q + k_1 w_0 - \frac{1}{3}k_2\varepsilon^2 w_0''\right) \tag{5.3}$$

On substituting this relation into the second equation of system (5.2), we obtain the equation

$$\frac{2}{3}\varepsilon^4 w_0^{IV}\left(1 + \frac{2+\nu}{4}k_2\right) - \left(k_2 + \frac{2+\nu}{2}k_1\right)\varepsilon^2 w_0'' + k_1 w_0 = -q - \frac{2+\nu}{2}\varepsilon^2 q''$$

in one unknown w_0.

Having determined w_0 from this, from Eqs (5.2) we calculate, for the known w_0, the remaining basic unknowns u_0, σ_{z0} and τ_0, in terms of which all the stresses and strains are calculated using straightforward operations on relations (2.6)–(2.8).

Note that, since $k_2 \ll 1$, the second term in the first brackets can be neglected. This gives

$$\frac{2}{3}\varepsilon^4 w_0^{IV} - \left(k_2 + \frac{2+\nu}{2}k_1\right)\varepsilon^2 w_0'' + k_1 w_0 = -q - \frac{2+\nu}{2}\varepsilon^2 q''$$

If the functions w_0 and q are functions of zero variability, that is, differentiation with respect to the argument x does not change their asymptotic order, the second term on the left-hand side and the second term on the right-hand side can be neglected as being small, and the equation is then identical to the classical equation for the flexure of a beam on an elastic base.

We write the third equation of system (5.2) in the following form in the unknown τ_0

$$(2+\nu)\varepsilon^2\tau_0'' - 2\tau_0 = \varepsilon^3 w_0''' - k_2(\varepsilon w_0' + u_0)$$

assuming that the functions w_0 and u_0 are known. We assume that the function w_0 and its derivatives are functions of zero variability. In accordance with this, we represent the final solution of this equation, written in the unknown τ_0 in the form of the sum of the particular solution τ_0^p and the general solution τ_0^g (when $w_0 = u_0 = 0$).

$$\tau_0^g = \begin{cases} C_1 e^{k(x-c)/\varepsilon} & \text{when } x \le c\\ C_2 e^{-k(x-c)/\varepsilon} & \text{when } x \ge c\end{cases}, \quad \tau_0^p = -\left\lfloor \varepsilon^3 w_0''' - k_2(\varepsilon w_0' + u_0)\right\rfloor/2$$

Substituting the expression

$$\tau_0 = -\left\lfloor \varepsilon^3 w_0''' - k_2(\varepsilon w_0' + u_0)\right\rfloor/2 + \tau_0^g$$

into Eq. (5.3), we obtain

$$\frac{2}{3}\varepsilon^4 w_0^{IV} + k_1 w_0 - \frac{4}{3}\varepsilon\tau_0^g - k_2\varepsilon^2 w_0'' - \frac{2}{3}k_2\varepsilon u_0' = -q$$

If we put $k_2 = \tau_0^g = 0$ in this equation, we obtain the classical equation for the flexure of a beam on an elastic base. When $k_2 = 0$, the equation

$$\frac{2}{3}\varepsilon^4 w_0^{IV} + k_1 w_0 - \frac{4}{3}\varepsilon\tau_0^{g\prime} = -q$$

differs from the classical equation in that it contains a singular term.

6. The convergence of the method

The required quantities, calculated in Section 2 by the method of simple iterations using relations (2.6), (2.7), (3.1) and (3.6), can be represented in the first approximation in the form of series in increasing powers of ε

$$w = w_0 + \nu\varepsilon^2 w_0'' z^2/2 - \varepsilon^4 (1+\nu)^2 \tau_0' z^2/2$$
$$u = \varepsilon[-w_0' z + \varepsilon^2 2(1+\nu)\tau_0 z], \ \ \varepsilon_x = \varepsilon^2[-w_0'' z + \varepsilon^2 2(1+\nu)\tau_0' z]$$
$$\sigma_x = \varepsilon^2[-w_0'' z + \varepsilon^2(2+\nu)\tau_0' z],$$
$$\varepsilon_z = \varepsilon^2[\nu w_0'' z - \varepsilon^2(1+\nu)^2\tau_0' z] \ \ \tau = \varepsilon^3[w_0''' z^2/2 + \tau_0 - \varepsilon^2(2+\nu)\tau_0'' z^2/2]$$
$$\sigma_z = -\varepsilon^4[w_0^{1V} z^3/6 + \tau_0' z - \varepsilon^2(2+\nu)\tau_0''' z^3/6]$$

Here, the components of the 0-process have not been taken into account since the problem of finding then is separate from the problem of determining of the components w_0 and τ_0.

We shall seek a solution of the initial Eq. (2.1) in accordance with these expressions, representing each of the required unknowns in the form of the sum of a slowly varying component and a rapidly varying component. Here, we write the slowly varying component in the form of an asymptotic series. For example, the displacement w is written as

$$w = w^s + \varepsilon^3 w^f = \sum_{n=0} w_n^s \varepsilon^{2n} + \varepsilon^3 w^f \tag{6.1}$$

The slowly varying component of the solution is labelled with the superscript s and the rapidly varying component with the superscript f. The remaining unknowns are written in a similar manner taking account of the asymptotic relations

$$u^s \sim \varepsilon w, \quad \varepsilon_x^s, \varepsilon_z^s, \sigma_x^s \sim \varepsilon^2 w, \quad \tau^s \sim \varepsilon^3 w, \ \sigma_z^s \sim \varepsilon^4 w,$$
$$w^f, u^f, \varepsilon_x^f, \varepsilon_z^f, \sigma_x^f, \tau^f, \sigma_z^f \sim \varepsilon^3 w$$

We substitute the expressions of the type (6.1) for all of the unknowns into the initial Eq. (2.1). This gives

$$\varepsilon \sum_{n=0} \frac{\partial u_n^s}{\partial z} + \varepsilon^3 \frac{\partial u^f}{\partial z} = -\varepsilon \sum_{n=0} \frac{\partial w_n^s}{\partial x} \quad + \varepsilon^3 + \sum_{n=0} 2(1+\nu)\tau_n^s - \varepsilon^6 \frac{\partial w^f}{\partial x} + \varepsilon^3 2(1+\nu)\tau^f$$
$$\varepsilon^4 \sum_{n=0} \frac{\partial \sigma_{zn}^s}{\partial z} + \varepsilon^3 \frac{\partial \sigma_z^f}{\partial z} = -\varepsilon^4 \sum_{n=0} \frac{\partial \tau_n^s}{\partial x} - \varepsilon^4 \frac{\partial \tau^f}{\partial x}$$
$$\varepsilon^2 \sum_{n=0} \varepsilon_{xn}^s + \varepsilon^3 \varepsilon_x^f = \varepsilon^2 \sum_{n=0} \frac{\partial u_n^s}{\partial x} + \varepsilon^4 \frac{\partial u^f}{\partial x}$$
$$\varepsilon^2 \sum_{n=0} \sigma_{xn}^s + \varepsilon^3 \sigma_x^f = \varepsilon^2 \sum_{n=0} \varepsilon_{xn}^2 + \varepsilon^4 \nu \sum_{n=0} \sigma_{zn}^s + \varepsilon^3 \varepsilon_x^f + \varepsilon^3 \nu \sigma_z^f$$
$$\varepsilon^2 \sum_{n=0} \varepsilon_{zn}^s + \varepsilon^3 \varepsilon_z^f = \varepsilon^4 (1-\nu^2) \sum_{n=0} \sigma_{zn}^s - \varepsilon^2 \nu \quad - \varepsilon^2 \nu \sum_{n=0} \varepsilon_{xn}^s + \varepsilon^3 (1-\nu^2) \sigma_z^f - \varepsilon^3 \nu \varepsilon_x^f$$
$$\sum_{n=0} \frac{\partial w_n^s}{\partial z} + \varepsilon^3 \frac{\partial w^f}{\partial z} = \varepsilon^2 \sum_{n=0} \varepsilon_{zn}^s + \varepsilon^3 \varepsilon_z^f$$
$$\varepsilon^3 \sum_{n=0} \frac{\partial \tau_n^s}{\partial z} + \varepsilon^3 \frac{\partial \tau^f}{\partial z} = -\varepsilon^3 \sum_{n=0} \frac{\partial \sigma_{xn}^s}{\partial x} - \varepsilon^4 \frac{\partial \sigma_x^f}{\partial x}$$

By individually equating the slowly varying and rapidly varying quantities on both side of the equalities, we obtain seven equations for determining the slowly varying quantities with a recursive form in order of increasing n

$$\frac{\partial u_n^s}{\partial z} = -\frac{\partial w_n^s}{\partial x} + 2(1+\nu)\tau_{n-1}^s, \quad \frac{\partial \sigma_{zn}^s}{\partial z} = -\frac{\partial \tau_n^s}{\partial x}, \quad \varepsilon_{xn}^s = \frac{\partial u_n^s}{\partial x}$$

$$\sigma^s_{xn} = \varepsilon^s_{xn} + \nu\sigma^s_{zn-1}, \quad \varepsilon^s_{zn} = (1-\nu^2)\sigma^s_{zn-1} - \nu\varepsilon^s_{xn},$$

$$\frac{\partial w^s_n}{\partial z} = \varepsilon^s_{zn-1}, \quad \frac{\partial \tau^s_n}{\partial z} = -\frac{\partial \sigma^s_{xn}}{\partial x}$$

where quantities with negative supscripts, obtained when $n=0$, are equal to zero, and seven equations for determining of the rapidly varying unknowns

$$\frac{\partial u^f}{\partial z} = -\varepsilon^2\frac{\partial w^f}{\partial \xi} + 2(1+\nu)\tau^f, \quad \frac{\partial \sigma^f_z}{\partial z} = -\frac{\partial \tau^f}{\partial \xi}$$

$$\varepsilon^f_x = \frac{\partial u^f}{\partial \xi}, \quad \sigma^f_x = \varepsilon^f_x + \nu\sigma^f_z, \quad \varepsilon^f_z = (1-\nu^2)\sigma^f_z - \nu\varepsilon^f_x$$

$$\frac{\partial w^f}{\partial z} = \varepsilon^f_z, \quad \frac{\partial \tau^f}{\partial z} = -\frac{\partial \sigma^f_x}{\partial \xi} \tag{6.2}$$

The substitution $\xi = x/\varepsilon$ has been made in these equations in order that differentiation with respect to ξ does not change the order of the differentiable quantities.

The first system of equation when $n=0$ is easily integrated:

$$w_0 = w_0(x), \quad u_0 = -w_0'z + \alpha$$

$$\varepsilon_{x0} = -w_0''z + \alpha', \quad \sigma_{x0} = \varepsilon_{x0}, \quad \varepsilon_{z0} = -\nu\varepsilon_{x0}$$

$$\tau_0 = w_0'''\frac{z^2}{2} - \alpha'z + \beta, \quad \sigma_{z0} = -w_0^{IV}\frac{z^3}{6} + \alpha''\frac{z^2}{2} - \beta'z + \gamma$$

Here, the superscript s is omitted for simplicity. The arbitrary functions of integration $\alpha(x)$, $\beta(x)$, $\gamma(x)$ correspond to the functions u_0, τ_0, f_{z0} in the solution (2.6)–(2.8). This solution corresponds to the solutions obtained by the method of simple iterations in the w-, τ- and 0-processes.

The term in the first equation of system (6.2) with the small factor ε^2 can be discarded, which considerably simplifies the equations and enables us to obtain the resolvent equation

$$\frac{\partial^2 \tau^f}{\partial z^2} + (2+\nu)\frac{\partial^2 \tau^f}{\partial \xi^2} = 0, \quad \frac{\partial^2 w^f}{\partial z^2} = -(1+\nu)^2\frac{\partial \tau^f}{\partial \xi}$$

The solution of the first equation, which satisfies the condition that there are no shear stresses on the longitudinal edges, is of the boundary layer type (or, more accurately, a superposition of boundary layers).

$$\tau^f = \begin{cases} \sum_{n=1}^{\infty} C_{2n}\exp\left(\kappa_n\frac{x-c}{\varepsilon}\right)\cos\frac{n\pi z}{2} \\ \sum_{n=1}^{\infty} C_{1n}\exp\left(-\kappa_n\frac{x-c}{\varepsilon}\right)\cos\frac{n\pi z}{2} \end{cases}, \quad \kappa_n = \frac{n\pi}{2(2+\nu)^{1/2}}$$

where C_{1n} and C_{2n} are constants of integration. The upper value is taken when $x \le c$ and the lower value when $x \ge c$.

Hence, we see that the solution of the initial problem of the theory of elasticity in the first approximation can be constructed by two routes. The first solution is obtained by the method of simple iterations. In the second solution, asymptotic series are constructed on the basis of asymptotic estimates obtained by the method of simple iterations and the calculations are then performed by the method of asymptotic integration. The second solution is the asymptotic solution and it satisfies the condition of asymptotic convergence, that is, the requirement that $\tau \to \tau_0$ when $\varepsilon \to 0$.

The two solutions are compared by means of an example. For a strip loaded with a concentrated load P at the point with the coordinates $x=c$, $z=1$, the shear stress τ, which satisfies the boundary conditions (2.9) and the matching conditions (3.7), calculated using Eq. (2.16), has the form

$$\tau^g = -\frac{3P}{8\varepsilon}(1-z^2) \times \begin{cases} (1-3c^2+2c^3)c\left[1-\exp\left(k\frac{x-c}{\varepsilon}\right)\right] \\ (-3+2c)c^3\left[1-\exp\left(-k\frac{x-c}{\varepsilon}\right)\right] \end{cases}$$

The upper value is taken when $x \le c$ and the lower value when $x \ge c$.

The same stress, calculated using expression (6.3), has the form

$$\tau^f = -\frac{12P}{\pi^3\varepsilon}\sum_{n=1}^{\infty}\frac{1}{n^3}\cos\frac{n\pi z}{2}\times\begin{cases}(1-3c^2+2c^3)c\left[1-\exp\left(\kappa_n\frac{x-c}{\varepsilon}\right)\right]\\(-3+2c)c^3\left[1-\exp\left(-\kappa_n\frac{x-c}{\varepsilon}\right)\right]\end{cases}$$

The upper value is taken when $x \le c$ and the lower value when $x \ge c$.

The calculations show that, when $c = 0.5$ m, the difference between the shear stresses obtained using the SI and SIAI methods does not exceed 5%. The model obtained using the SI method corresponds to the case when only the first term in the expansion in a Fourier series is taken into account in the more accurate and logically consistent model obtained using the SIAI method. It is customarily assumed that this corresponds to models of the strength of materials.[8]

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